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An Alternative Form of the Equation of Motion for Constrained Structural and Mechanical Systems With Singular Mass Matrices

This paper develops an alternative description of the general equation of motion for constrained mechanical systems with singular mass matrices. The formulation gives a new explicit equation of motion for such systems and provides a simple and elegant way to interpret the manner in which Nature orchestrates constrained motion, something that was not possible for such systems in the past. [DOI: 10.1115/1.4056293]

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1 Introduction

A central problem in analytical dynamics is the determination of the equation of motion for constrained mechanical systems. While the problem was first formulated and broached by Lagrange [1] more than 225 years ago it has since been continuously and actively pursued by numerous scientists, mathematicians, and engineers. Lagrange specifically invented the method of Lagrange multipliers to handle constrained motion and provided an approach to determine the equations of motion for holonomic systems. Gauss [2] developed a new underlying principle in mechanics for handling constrained motion, known today as Gauss's Principle. It is equivalent to d'Alembert's Principle, which was first enunciated in its exact form, it appears, by Lagrange. The Gibbs–Appell approach to the description of constrained motion was independently discovered by Gibbs [3] and Appell [4], and Dirac developed a recursive scheme using Poisson brackets for singular Hamiltonian systems [5]. Obtaining the equation of motion for mechanical systems subjected to sets of nonholonomic constraints was especially difficult, and general results encompassing general holonomic and nonholonomic constraints were unavailable. While the method of Lagrange multipliers can be used, it becomes unfeasible when dealing with large dimensional systems subjected to many nonholonomic constraints. The difficulty in obtaining a general explicit equation of motion for systems that have general holonomic and/or nonholonomic constraints is highlighted in Ref. [6], a gold standard in classical mechanics, where it is stated early on in the book that "But there is no general way to attack nonholonomic examples … the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics, it is almost invariably assumed that any constraint, if present, is holonomic." It should be noted that the above-mentioned investigations focus on nonholonomic constraints that are in the so-called Pfaffian form, meaning that the nonholonomic constraints are linear in the generalized velocities [6–8].

Udwadia and Kalaba [9,10] discovered a simple explicit equation of motion for general constrained mechanical systems that have holonomic and/or nonholonomic constraints in which the constraints are not necessarily Pfaffian in form and can be any (consistent) set of nonlinear functions of the (generalized) coordinates, velocities, and time. Furthermore, the constraints need not be functionally independent. This equation is referred to as the Fundamental Equation of Constrained Motion (FECM) due to its applicability to systems in which the constraints are, in general, holonomic and/or nonholonomic. A significant advantage of this equation is that it provides physical insights into the simplicity with which Nature seems to execute constrained motion, showing that she seems to behave much like a control engineer engaged in feedback control. All the aforementioned investigators used d'Alembert's principle (assumption), and/or, equivalently Gauss's principle, as their starting point [6]. D'Alembert's principle states that the total work done by all the forces of constraint under all virtual displacements adds up to zero. While this assumption, which is at the core of much of analytical dynamics, seems to work well in many practical situations, there are equally many instances, in Nature and in engineered systems, where this assumption may not hold. Most often, this arises when energy is extracted at a constraint, for example, when damping may be present, or when it is injected into the system as in maglev trains. Generalized equations for constrained systems that include constraints that may or may not satisfy d'Alembert's principle (assumption) were obtained later [11,12].

The investigations reported in the previous paragraphs all assume that the mass matrix of the dynamical system is positive definite. When the minimum number of coordinates are used for modeling an unconstrained mechanical system, the mass matrix in the Lagrange equations is usually positive definite [13]. However, there are many practical situations when the mass matrix may be singular. A further generalization of the equation of motion for constrained systems that may have singular mass matrices was obtained by Udwadia and Phohomsiri [14]. The structure of the equation of motion obtained, though, was quite different from the FECM's structure when the mass matrix is positive definite. To remedy this, Udwadia and Schutte [15] developed an equivalent equation of motion for constrained systems with singular mass matrices that has the same structure as the FECM in which the mass matrix is positive definite. Improvements were made by Udwadia

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and Wanichanon [16] that further simplified and generalized this equation and improved its computational efficiency.

Mechanical systems with singular mass matrices occur most frequently when more (generalized) coordinates are used in describing a mechanical system than the minimum number required so that the coordinates are not in fact independent of one another and are subjected to constraints. Perhaps, the most common example of this is the use of quaternions to describe the rotational motion of a rigid body. The three Euler angles provide the minimum number of independent coordinates (parameters) needed to describe the orientation of a rigid body. Quaternions are used to avert singularities in the determination of the angular velocity of a rigid body, a problem that is difficult to avert when Euler angles are used to describe rotations. However, quaternions are described by four parameters. Since only three independent parameters are needed to describe the rotation of a rigid body, the four quaternion parameters are therefore not independent and are related to one another through the constraint that their Euclidean norm is unity. And hence, the rotational motion of a rigid body described by quaternions leads to a singular mass matrix. There are also many other instances, when the use of more coordinates than the minimum number required becomes very helpful in modeling complex systems, such as, in facilitating the determination of their equations of motion and/or obtaining the equations in a more computationally efficient form [17]. Other instances arise when coordinates need to be added at locations in complex systems that underpin (and/or make more comprehensible) their dynamical behavior [18]. Often, one may be interested in decomposing a complex multi-body system into its constituent components for each of which the equations of motion are known [14]. One may then want to use these equations for the components to synthesize the equations of motion of the composite system. Singular mass matrices can arise in such circumstances too. Thus, singular mass matrices also arise when one wants more flexibility in modeling complex mechanical systems.

In this paper, we develop a new general and explicit equation of motion for systems that may or may not have singular mass matrices. It is applicable to systems with holonomic and/or nonholonomic constraints that may or may not be functionally independent, as well as systems that do or do not satisfy d'Alembert's assumption. The equation provides useful insights, so far unavailable, into the simplicity and aesthetics with which Nature executes constrained motion of mechanical and structural systems that may have singular mass matrices. We also show that it reduces to the known FECM when the mass matrix is restricted to being positive definite.

To introduce the notation and place the problem of constrained motion in context, we begin by considering the motion of an unconstrained system at any time t given by the Lagrange equation

$$M(q, t)a(t) = Q(q, \dot{q}, t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0 \tag{1}$$

where q is the generalized coordinate n-vector, Q is the (known) "given" force which is a function of q, $\dot{q}$, and t, a is the acceleration n-vector of the unconstrained system, and n is the number of generalized coordinates. We shall assume that the mass matrix M is a symmetric n-by-n matrix which in general is positive semidefinite; it can therefore be singular. By "unconstrained" we mean that the n coordinates, q, are independent of one another.

We next assume that this unconstrained system is subjected to a set of m consistent constraints

$$\varphi_i(q, \dot{q}, t) = 0, \quad i = 1, 2, \ldots, m \tag{2}$$

and the initial conditions stated in Eq. (1) satisfy them. These constraints include both holonomic and nonholonomic constraints, and then some. The functions φ_i can be functionally dependent. Our aim is to determine the acceleration $\ddot{q}(t)$ of the dynamical system described by Eq. (1) in the presence of the constraints described by Eq. (2).

Under the assumption that the constraints are C^1 functions, we differentiate them with respect to time to obtain the relation

$$A(q, q, t)\ddot{q} = b(q, \dot{q}, t) \tag{3}$$

where A is an m-by-n matrix whose rank is $k \le m$, and b is an m-vector. It should be noted that for a given set of initial conditions, Eq. (2) is equivalent to Eq. (3). We will refer to A as the constraint matrix.

The constraints imposed on the system described by Eq. (1) give rise to an additional force n-vector, Q^c, called the (generalized) force of constraint, that acts on it so that the equation of motion of the constrained system now becomes

$$M(q, t)\ddot{q} = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0 \tag{4}$$

in which the n-by-n matrix M can be singular, as stated before. The n-vector $\ddot{q}$ gives the acceleration of the constrained system, whose explicit form we aim to obtain in Sec. 2.

According to d'Alembert's assumption, the force of constraint Q^c, which is usually used in classical mechanics, assumes that at every instant of time t, the work done by Q^c under any (non-zero) virtual displacement, $w(t)$, is zero. A virtual displacement n-vector at time t is any vector that lies in the null space of the matrix A, $\mathbb{N}(A)$, and therefore satisfies the relation [19]

$$A(q(t), \dot{q}(t), t)w(t) = 0 \tag{5}$$

We point out that since Eq. (5) is homogeneous, if any virtual displacement n-vector, $w \neq 0$, satisfies it, then so does the n-vector αw, for *any* scalar $\alpha \neq 0$, and hence, a virtual displacement need not be infinitesimal in magnitude.

When the (generalized) force of constraint Q^c satisfies d'Alembert's assumption so that $w^T Q^c = 0$ at every instant of time, it is referred to as an ideal constraint (force). However, there are many systems in which the force of constraint *does* do work, and therefore, we assume here that the work done by the force of constraint under virtual displacements can be, in general, positive, negative, or zero. Thus,

$$w^T Q^c(q, \dot{q}, t) = w^T C(q, \dot{q}, t) \tag{6}$$

where the given n-vector C describes the nature of the non-ideal constraint force that is acting and depends of course on the specific system under consideration [11,12]. For brevity, from here on, we shall suppress the arguments of the various quantities in the equations, unless required for clarity.

Since the solution of Eq. (5) at each instant of time t is

$$w = (I - A^+A)v \tag{7}$$

where $A^+(q, \dot{q}, t)$ is the Moore–Penrose (MP) inverse of A and v is any *arbitrary* n-vector, using Eq. (7) in Eq. (6) yields

$$v^T(I - A^+A)Q^c(q, \dot{q}, t) = v^T(I - A^+A)C(q, \dot{q}, t) \tag{8}$$

from which it follows that

$$(I - A^+A)Q^c = (I - A^+A)C \tag{9}$$

Premultiplying both sides of Eq. (4) by $(I - A^+ A)$, we obtain

$$(I - A^+A)M\ddot{q} = (I - A^+A)(Q + C) \tag{10}$$

We are now ready to obtain an alternative form of the Fundamental Equation of Constrained Motion (FEMC).

2 Alternative Form of the Fundamental Equation of Constrained Motion

In this section, we obtain the FECM and provide an interpretation of the manner in which Nature contrives it. We consider in detail the various terms that appear in the equation and provide additional geometric and algebraic insights into them.

We note that a given constrained mechanical or structural system is fully specified by its n by n matrix $M \geq 0$, its m by n matrix A, the given n-vectors Q, C, and the given m-vector b. By specified we mean here that these entities are known functions of q, $\dot{q}$, and t.

2.1 Determination of the Fundamental Equation of Constrained Motion. The acceleration n-vector $\ddot{q}$ of the constrained system in Eq. (4) can be described in terms of its orthogonal projections on $\mathbb{N}(A)$ and $\mathbb{N}(A)^{\perp} = \mathbb{R}(A^T)$ as

$$\ddot{q} = (I - A^+A)\ddot{q} + A^+A\ddot{q} := \ddot{q}_{\mathbb{N}(A)} + \ddot{q}_{\mathbb{N}(A)^{\perp}} \tag{11}$$

The first member on the right in the first equality is the projection of $\ddot{q}$ on $\mathbb{N}(A)$, denoted by $\ddot{q}_{\mathbb{N}(A)}$, and the second is the projection on its orthogonal complement, denoted by $\ddot{q}_{\mathbb{N}(A)^{\perp}}$. Since the acceleration $\ddot{q}$ of the constrained system must satisfy the constraint equations, and therefore must satisfy Eq. (3), the first equality in the last equation can be rewritten as

$$\ddot{q} = (I - A^+A)\ddot{q} + A^+b \tag{12}$$

Thus, the projection $\ddot{q}_{\mathbb{N}(A)^{\perp}}$ of the acceleration $\ddot{q}$ is trivially obtained; it is $A^+ b$. Our main task now is to obtain $\ddot{q}_{\mathbb{N}(A)} = (I - A^+A)\ddot{q}$. Using Eq. (12) in Eq. (10), we get

$$(I - A^+A)M(I - A^+A)\ddot{q} = (I - A^+A)(Q + C) - (I - A^+A)MA^+b \tag{13}$$

Equations (3) and (13) can be rewritten as

$$\tilde{M}\ddot{q} := \begin{bmatrix} M_s \\ A \end{bmatrix}\ddot{q} = \begin{bmatrix} (I - A^+A)(Q + C) - (I - A^+A)MA^+b \\ b \end{bmatrix} \tag{14}$$

where we have denoted the symmetric n-by-n matrix

$$M_s := (I - A^+A)M(I - A^+A), \quad M \geq 0 \tag{15}$$

and the $(m+n)$-by-n matrix $\tilde{M}$ as

$$\tilde{M} := \begin{bmatrix} M_s \\ A \end{bmatrix} = \begin{bmatrix} (I - A^+A)M(I - A^+A) \\ A \end{bmatrix}, \quad M \geq 0 \tag{16}$$

The subscript "s" in Eq. (15) signifies that the matrix M_s is symmetric.

The solution of Eq. (14) is given by

$$\ddot{q} = \tilde{M}^+\begin{bmatrix} (I - A^+A)(Q + C) - (I - A^+A)MA^+b \\ b \end{bmatrix} + (I - \tilde{M}^+\tilde{M})\gamma \tag{17}$$

where γ is an arbitrary n-vector.

We observe that, in general, the acceleration of the constrained system is not necessarily unique because of the second member on the right-hand side of Eq. (17). However, when the $(m+n)$ by n matrix $\tilde{M}$ has full column rank, this second member vanishes, because then $\tilde{M}^+ = (\tilde{M}^T\tilde{M})^{-1}\tilde{M}^T$ so that $\tilde{M}^+\tilde{M} = I$, and $\ddot{q}$ becomes unique. Thus, when the matrix $\tilde{M}$ has full column rank, the acceleration of the constrained mechanical system is given by

$$\ddot{q} = \tilde{M}^+\begin{bmatrix} (I - A^+A)(Q + C) - (I - A^+A)MA^+b) \\ b \end{bmatrix} \tag{18}$$

Remark 1. In classical mechanics the acceleration vector that is experimentally observed is unique. We therefore expect our theoretical models to also deliver unique accelerations. Thus, the rank of $\tilde{M}$ can be used as a check to assess the adequacy of a given method of modeling for a mechanical or structural system. ■

Our next task is to find the Moore–Penrose (MP) inverse of the matrix $\tilde{M}$ and simplify the right-hand side of Eq. (18). In order to do this, we use the following two Lemmas.

LEMMA 1.

(a) If $R = \begin{bmatrix} S \\ T \end{bmatrix}$ and $ST^T = 0$, then $R^+ = [S^+ \mid T^+]$ and $R^+ R = S^+ S + T^+ T$.

(b) When the matrix R described earlier has, in addition, full column rank, then $I = S^+ S + T^+ T$.

Proof.

(a) This can be proved by directly showing that R^+ satisfies the four MP (Moore–Penrose) conditions. The result is well-known (e.g., see Ref. [19, Theorem 6.4.5]).

(b) When R has, in addition, full column rank, R^TR is nonsingular and positive definite. Hence, $R^+ = (R^TR)^+ R^T = (R^TR)^{-1}R^T$, so that $R^+ R = I$. Using this in Part (a) of this lemma, the result follows. ■

LEMMA 2.

$$\begin{aligned} &\text{(a)} \quad M_s = M_s(I - A^+A) = (I - A^+A)M_s \\ &\text{(b)} \quad M_s^+ = M_s^+(I - A^+A) = (I - A^+A)M_s^+ \end{aligned} \tag{19}$$

$$\text{(c)} \quad M_s^+ \text{ is symmetric} \tag{20}$$

Proof.

(a) Postmultiplying both sides of Eq. (15) by $(I - A^+ A)$, we get

$$\begin{aligned} M_s(I - A^+A) &= (I - A^+A)M(I - A^+A)(I - A^+A) \\ &= (I - A^+A)M(I - A^+A) = M_s \end{aligned} \tag{21}$$

where the second equality above follows because $(I - A^+ A)$ is idempotent. Similarly, by premultiplying both sides of Eq. (15) by $(I - A^+ A)$, the second equality in Part (a) of Eq. (19) follows.

(b) The matrix M_s^+ can always be written as

$$M_s^+ = (M_s^TM_s)^+M_s^T = (M_sM_s)^+M_s \tag{22}$$

Postmultiplication by $(I - A^+ A)$ then yields

$$\begin{aligned} M_s^+(I - A^+A) &= (M_sM_s)^+M_s(I - A^+A) \\ &= (M_sM_s)^+M_s = M_s^+ \end{aligned} \tag{23}$$

where the second equality above follows from the first equality in Part (a), and the last equality follows from Eq. (22).

Similarly, writing $M_s^+ = M_s^T(M_sM_s^T)^+ = M_s(M_sM_s)^+$, and premultiplying both sides of this relation by $(I - A^+ A)$ and using the second equality in Part (a) gives the second equality in Part (b) of Eq. (19).

(c) We note that M_s^+ is a symmetric n by n matrix since $[M_s^+]^T = [M_s^T]^+ = M_s^+$. ■

Result 1

When $\tilde{M}$ has full column rank, the acceleration of the constrained mechanical system is explicitly given by

$$\ddot{q} = M_s^+(Q + C - MA^+b) + A^+b := M_s^+\Delta + A^+b \tag{24}$$

Proof. The matrix

$$\begin{aligned} M_sA^T &= (I - A^+A)M(I - A^+A)A^T \\ &= (I - A^+A)M(I - (A^+A)^T)A^T \\ &= (I - A^+A)M[I - A^T(A^+)^T]A^T \\ &= (I - A^+A)M[I - A^T(A^T)^+]A^T = 0 \end{aligned} \tag{25}$$

The second equality follows from the fourth MP condition, and the last follows from the first MP condition. Hence, the rows of the matrix M_s (Eq. (15)) are orthogonal to the rows of the constraint matrix A (Eq. (3)).

Since $\tilde{M} = \begin{bmatrix} M_s \\ A \end{bmatrix}$, using Lemma 1, Part (a), we obtain

$$\tilde{M}^+ = [M_s^+ | A^+] \tag{26}$$

and Eq. (18) becomes

$$\begin{aligned} \ddot{q} &= [M_s^+ | A^+] \begin{bmatrix} (I - A^+A)(Q + C) - (I - A^+A)MA^+b \\ b \end{bmatrix} \\ &= M_s^+(I - A^+A)(Q + C) - M_s^+(I - A^+A)MA^+b + A^+b \\ &= M_s^+(Q + C - MA^+b) + A^+b = M_s^+\Delta + A^+b, \end{aligned} \tag{27}$$

where $\Delta = Q + C - MA^+b$. The third equality follows from Lemma 2, Part (b). ■

To illustrate the use of Eq. (24) consider the special case of a particle of mass $m_0 > 0$ moving in three-dimensional configuration space that is subjected to a given force 3-vector, $Q(q, \dot{q}, t)$, so that the unconstrained motion of m_0 is described by the relation

$$m_0 I_3 a = m_0 a = Q(q, \dot{q}, t)$$

where the 3-vector a is the acceleration of the unconstrained system. The given 3-vector $C(q, \dot{q}, t)$ can similarly be written as $m_0 d = C(q, \dot{q}, t)$. Assume that the particle is subjected to holonomic and/or nonholonomic constraints described by the equation $A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t)$.

The matrix $M_s = m_0(I - A^+ A)(I - A^+ A) = m_0(I - A^+ A)$, so that $M_s^+ = (1/m_0)(I - A^+A)$. Equation (24) then gives the explicit equation of motion of the particle under general constraints as

$$\begin{aligned} \ddot{q} &= (1/m_0)(I - A^+A)[m_0 a + m_0 d - m_0 IA^+b] + A^+b \\ &= (I - A^+A)(a + d - A^+b) + A^+b \end{aligned}$$

Noting that $(I - A^+ A)A^+ = 0$, the equation of motion of the constrained system is then

$$\ddot{q} = (I - A^+A)(a + d) + A^+b \tag{28}$$

The striking simplicity of this equation has a simple and elegant geometrical interpretation. The 3-vector $\ddot{q}$ is made up of two orthogonal components: the projection of $\ddot{q}$ on $\mathbb{N}(A)$ is the projection of the vector $(a + d)$ on $\mathbb{N}(A)$, and the projection of the $\ddot{q}$ on $\mathbb{N}(A)^\perp$ is $A^+ b$. Figure 1(*a*) on the left shows the geometric interpretation of Eq. (28) pictorially. The point $O = \{0\}$.

Figure 1(*b*) on the right principally deals with the forces acting on the system and shows the geometry in terms of these forces. The figure shows the n-vector Δ for which $\Delta + MA^+ b = Q + C$, and the n-vectors $Q = m_0 a$ and $C = m_0 d$. Since $(I - A^+ A)MA^+ b = m_0(I - A^+ A)A^+ b = 0$, the projection of the n-vector $MA^+ b$ on $\mathbb{N}(A)$ is zero. Thus, the n-vector $MA^+ b$ belongs to $\mathbb{N}(A)^\perp$ and is orthogonal (perpendicular) to $\mathbb{N}(A)$, as shown. Consequently, the projections of the n-vectors Δ and $(Q + C)$ on $\mathbb{N}(A)$ are identical and the figure shows that $(I - A^+ A)\Delta = (I - A^+ A)(Q + C)$. The n-vector $\ddot{q}_{\mathbb{N}(A)} = M_s^+\Delta = (I - A^+A)\Delta/m_0 = (I - A^+A)(Q + C)/m_0 = (I - A^+A)(a + d)$, as also seen in Fig. 1(*a*), which concentrates only on the components of the acceleration $\ddot{q}$ of the constrained system. Lastly, $\ddot{q}_{\mathbb{N}(A)^\perp} = A^+b$.

Remark 2. More generally, the first member, $M_s^+\Delta$, on the right-hand side of Eq. (24) is simply the orthogonal projection of the acceleration of the constrained system, $\ddot{q}$ on $\mathbb{N}(A)$, since

$$\begin{aligned} \ddot{q}_{\mathbb{N}(A)} &= (I - A^+A)\ddot{q} = (I - A^+A)[M_s^+\Delta + A^+b] \\ &= (I - A^+A)M_s^+\Delta = M_s^+\Delta = M_s^+(Q + C - MA^+b) \end{aligned} \tag{29}$$

In the third equality, we have used the fact that $(I - A^+ A)A^+ = 0$, and in the fourth equality, we have used Lemma 2, Part (b). Similarly, the orthogonal projection on the subspace $\mathbb{N}(A)^\perp = \mathbb{R}(A^T)$ is

$$\begin{aligned} \ddot{q}_{\mathbb{N}(A)^\perp} &= A^+A\ddot{q} = A^+A[M_s^+\Delta + A^+b] \\ &= A^+A[(I - A^+A)M_s^+\Delta] + A^+AA^+b \\ &= A^+b \end{aligned} \tag{30}$$

which is something we already knew. In the last equality above, we have used the second MP condition, $A^+ AA^+ = A^+$.

The geometric interpretation obtained from Eqs. (29) and (30), where the n-vector Δ is such that $\Delta + MA^+ b = Q + C$, $\ddot{q} = M_s^+\Delta + A^+b$, and $M \geq 0$, is depicted in Fig 2. It is a generalization of Fig. 1(*b*). As before, the projection of the acceleration n-vector $\ddot{q}$ on $\mathbb{N}(A)^\perp$ is $A^+ b$. However, the orthogonal projection of the n-vector Δ on $\mathbb{N}(A)$, which is $(I - A^+ A)\Delta$, no longer, in general, equals the projection of the n-vector $(Q + C)$ on $\mathbb{N}(A)$.

This projection, as shown in Fig. 2, is an n-vector from O to the open circle along the $\mathbb{N}(A)$ axis. The mapping M_s^+ maps this vector to one from O to the solid circle (also) along the $\mathbb{N}(A)$ axis, because by Lemma 2, Part (b), $M_s^+(I - A^+A)\Delta = M_s^+\Delta = \ddot{q}_{\mathbb{N}(A)}$ (Eq. (21)). Hence, $\|\ddot{q}_{\mathbb{N}(A)}\| \leq \|M_s^+\|\|\Delta\|$, while the magnitude of the component $\ddot{q}_{\mathbb{N}(A)^\perp} = A^+b$ is $\|\ddot{q}_{N(A)^\perp}\| \geq \|MA^+b\|/\|M\|$.

We observe that the major difference between the special case described in Fig. 1(*b*) and the general situation shown in Fig. 2 is that the n-vector $MA^+ b$ in Fig. 1(*b*) has no projection on $\mathbb{N}(A)$ and hence belongs entirely to $\mathbb{N}(A)^\perp$, while in Fig. 2 this does not happen and $MA^+ b$ has, in general, a (nonzero) component in $\mathbb{N}(A)$. As mentioned before, this causes the projection of the

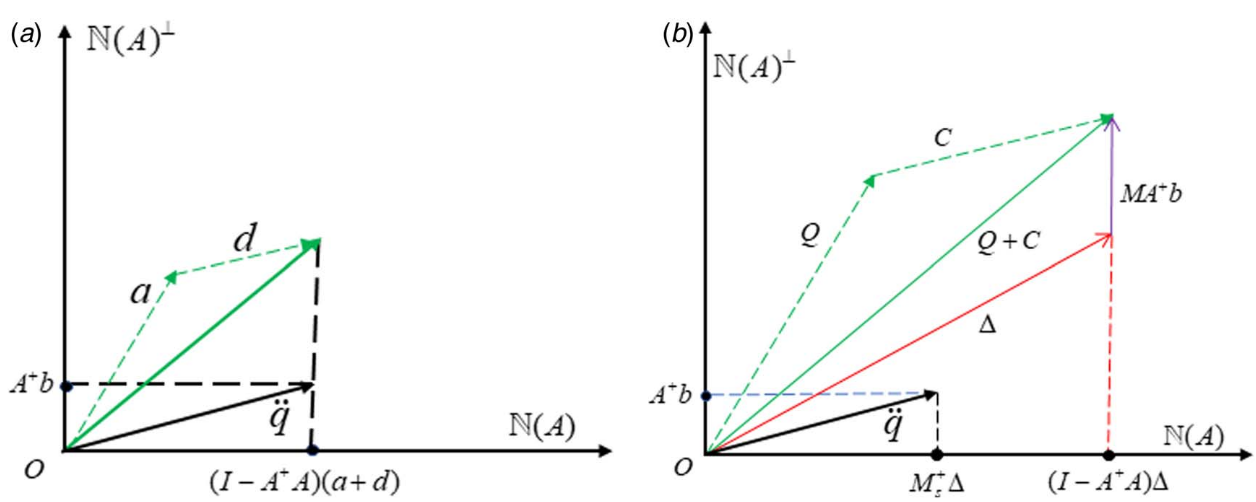

Fig. 1 Orthogonal components of: (*a*) $\ddot{q}$ in Eq. (28); (*b*) $\ddot{q}$ in Eq. (27)

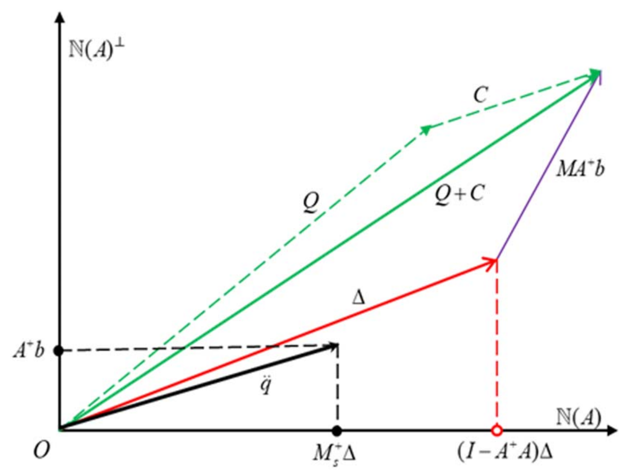

Fig. 2 Geometrical portrait of generalized forces and their projections on $\mathbb{N}(A)$ and $\mathbb{N}(A)^{\perp}$ that result in the explicit determination of the acceleration $\ddot{q}$ of the constrained system

n-vector Δ to be identical to projection of the n-vector $(Q+C)$ in the special case, as shown in Fig. 1(*b*), thereby providing a simpler geometrical portrait. ■

Having found $\ddot{q}$ explicitly using Eq. (24), we now determine the (generalized) constraint force, Q^c explicitly.

Result 2

When $\tilde{M}$ has full column rank, the force of constraint Q^c is explicitly given by

$$Q^c = M\ddot{q} - Q = M[M_s^+(Q + C - MA^+b) + A^+b] - Q \tag{31}$$

which can also be written as

$$Q^c = (MM_s^+ - I)(Q + C - MA^+b) + C = (MM_s^+ - I)\Delta + C \tag{32}$$

The orthogonal projections (components) of Q^c on $\mathbb{N}(A)^{\perp}$ and $\mathbb{N}(A)$ are given, respectively, by

$$Q^c_{\mathbb{N}(A)^{\perp}} = (MM_s^+ - I)(Q + C - MA^+b) + A^+AC = (MM_s^+ - I)\Delta + A^+AC \tag{33}$$

and

$$Q^c_{\mathbb{N}(A)} = (I - A^+A)C \tag{34}$$

Proof. Using Eqs. (4) and (24), Eq. (31) follows. Equation (32) also follows because

$$\begin{aligned} Q^c &= M[M_s^+(Q + C - MA^+b) + A^+b] - Q \\ &= (MM_s^+ - I)Q + MM_s^+C - (MM_s^+ - I)MA^+b \\ &= (MM_s^+ - I)(Q - MA^+b) + MM_s^+C \\ &= (MM_s^+ - I)(Q + C - MA^+b) + C = (MM_s^+ - I)\Delta + C \end{aligned} \tag{35}$$

We next show that the first member on the right-hand side of Eq. (32), namely $(MM_s^+ - I)\Delta$, belongs to $\mathbb{N}(A)^{\perp}$. Using Eqs. (9) and (32), we find that

$$0 = (I - A^+A)(Q^c - C) = (I - A^+A)[(MM_s^+ - I)\Delta] \tag{36}$$

This shows that the orthogonal projection of $(MM_s^+ - I)\Delta$ on $\mathbb{N}(A)$ is zero. Thus, the n-vector $(MM_s^+ - I)\Delta$ belongs to $\mathbb{N}(A)^{\perp}$, and so its orthogonal projection on $\mathbb{N}(A)^{\perp}$ is, of course, itself.

Now, using Eq. (32) and taking the orthogonal projection of Q^c on $\mathbb{N}(A)^{\perp}$, we therefore get

$$\begin{aligned} Q^c_{\mathbb{N}(A)^{\perp}} &:= A^+AQ^c = A^+A[(MM_s^+ - I)\Delta] + A^+AC \\ &= (MM_s^+ - I)\Delta + A^+AC \end{aligned} \tag{37}$$

where we have made use of the fact that the projection on $\mathbb{N}(A)^{\perp}$ of the n-vector $(MM_s^+ - I)\Delta$ (which belongs to $\mathbb{N}(A)^{\perp}$) is itself.

Also, from Eq. (9), the projection of Q^c on $\mathbb{N}(A)$ is given by

$$Q^c_{\mathbb{N}(A)} := (I - A^+A)Q^c = (I - A^+A)C \tag{38}$$

where C is a given n-vector for the specific system being modeled. The sum of Eqs. (37) and (38) yields Q^c, which is given by Eq. (32). ■

Remark 3. If the m by n constraint matrix A has full column rank n, then, $(I - A^+A) = 0$, and $\mathbb{N}(A) = \{0\}$. Hence, $\ddot{q}_{\mathbb{N}(A)} = 0$, and the first member on the right-hand side of Eq. (24) is zero (Eq. (29), $M_s^+ = 0$). Hence,

$$\ddot{q} = \ddot{q}_{\mathbb{N}(A)^{\perp}} = A^+b \tag{39}$$

Equation (39) shows that the acceleration of the constrained system, $\ddot{q}$, is now solely (and uniquely) dictated by the set of constraints to which the system is subjected, and the acceleration $\ddot{q}$ solves the equation $A\ddot{q} = b$ in the minimum-norm least squares sense. The constraint force is then obtained using Eqs. (31) and (39) as

$$Q^c = M\ddot{q} - Q = MA^+b - Q \tag{40}$$

■

Remark 4. The condition that the matrix $\tilde{M}$ has full column rank in Result 1 implies that it maps n-vectors $\ddot{q}$ in a 1-to-1 manner; that is, two different n-vectors are never mapped to the same vector in $\mathfrak{R}^{(m+n)}$. ■

Remark 5. The matrix (mapping) M_s, which is defined in Eq. (15), appears prominently in Eq. (24). To provide insight into this mapping, we consider its geometric and its algebraic descriptions.

(a) Geometric description of M_s: To understand the mapping M_s, we consider its effect on an arbitrary n-vector w. The orthogonal projection of w on $\mathbb{N}(A)$ is $w_{\mathbb{N}(A)} = (I - A^+A)w$, and the orthogonal projection of the matrix M on the null space $\mathbb{N}(A)$ is $M_{\mathbb{N}(A)} = (I - A^+A)M$. We then find that

$$\begin{aligned} M_{\mathbb{N}(A)} \underbrace{(I - A^+A)w}_{w_{\mathbb{N}(A)}} &= (I - A^+A)M(I - A^+A)w \\ &= \underbrace{(I - A^+A)M(I - A^+A)}_{M_s} \underbrace{(I - A^+A)w}_{w_{\mathbb{N}(A)}} = M_s w_{\mathbb{N}(A)} \end{aligned} \tag{41}$$

where we have used the idempotence of the matrix $(I - A^+A)$ in the second equality. Similarly, for any n-vector $w \in \mathfrak{R}^n$, $M_s w$ can be written as

$$M_s w = M_s(I - A^+A)w = M_s w_{\mathbb{N}(A)} = M_{\mathbb{N}(A)} w_{\mathbb{N}(A)} \tag{42}$$

Equation (42) says that the mapping M_s applied to any vector $w \in \mathfrak{R}^n$ is the same as though it were applied to the component of $w \in N(A)$, i.e., $w_{\mathbb{N}(A)}$; this is equivalent to the mapping $M_{\mathbb{N}(A)}$ applied to $w_{\mathbb{N}(A)}$. In what follows, the vector w in Eq. (42) will be particularized to the acceleration vector $\ddot{q}$ of the constrained system.

(b) Algebraic description of M_s: Consider the standard singular value decomposition of the m by n matrix A given by

$$A = U\Sigma V^T = [U_1|U_2]\,\Sigma\,[V_1|V_2]^T \tag{43}$$

where Σ is the m by n diagonal matrix containing the k singular values of A, and U and V are m-by-m and n-by-n orthogonal matrices, respectively. The matrix U_1 is m by k, and V_1 is n by k, since the rank of A is k. The n-by-$(n-k)$ matrix V_2 spans the null space of A, and the projection matrix $(I - A^+A) = (I - V_1V_1^T) = V_2V_2^T$ projects an arbitrary

n-vector, w, on to $\mathbb{N}(A)$. Hence

$$M_s w = (I - A^+ A) M (I - A^+ A) w = V_2 V_2^T M V_2 V_2^T w := V_2 \widehat{M} V_2^T w \tag{44}$$

where we have defined the symmetric $(n-k)$-by-$(n-k)$ matrix $\widehat{M} := V_2^T M V_2$. The last equality in Eq. (44) shows again that $M_s w$ belongs to $\mathbb{N}(A)$. Since w is an arbitrary n-vector, Eq. (44) implies that

$$M_s = V_2 \widehat{M} V_2^T \tag{45}$$

■

2.2 Interpretation of the Fundamental Equation of Constrained Motion. According to Eq. (24), at each instant of time Nature appears to find the two orthogonal projections (components), $\ddot{q}_{\mathbb{N}(A)^\perp}$ and $\ddot{q}_{\mathbb{N}(A)}$, of the acceleration $\ddot{q}$ of the constrained system by solving *linear* equations and *always* using their *minimum-norm least squares solutions*. At each instant of time, she appears to determine the acceleration of the constrained motion in two *sequential* steps.

(1) Nature appears first and foremost to satisfy the constraint $A\ddot{q} = b$ imposed on the dynamical system. She obtains the minimum-norm (Euclidean) least squares solution to this equation, which is $\ddot{q} = A^+ b$. This solution turns out to be $\ddot{q}_{\mathbb{N}(A)^\perp}$ (Eq. (30)).

(2) Nature then appears to consider an *auxiliary* system with the special mass matrix M_s, whose interpretation is given in Remark 5. The auxiliary system's mass matrix is subjected to (i) the prescribed n-vector C for the specific system under consideration, and (ii) the given (generalized) force n-vector Q (to which the unconstrained system is subjected, see Eq. (1)) from which the force contribution provided by the acceleration $\ddot{q}_{\mathbb{N}(A)^\perp}$, namely $M\ddot{q}_{\mathbb{N}(A)^\perp} = MA^+ b$, is removed. She uses here the $\ddot{q}_{\mathbb{N}(A)^\perp}$ that she has already found in Step (1).

The auxiliary system, whose acceleration is denoted by $\ddot{q}_A$, is thus described by the relation

$$M_s \ddot{q}_A = C + (Q - MA^+ b) \tag{46}$$

Considering this equation as a linear equation in $\ddot{q}_A$, Nature again uses the minimum-norm least squares solution given by

$$\ddot{q}_A = M_s^+ (Q - MA^+ b + C) \tag{47}$$

This solution $\ddot{q}_A$ turns out to be $\ddot{q}_{\mathbb{N}(A)}$ (Eq. (29)).

Notice that the right-hand side of Eq. (46) is not necessarily in $\mathbb{N}(A)$. It is the special character of matrix M_s chosen by Nature—for which $(I - A^+ A) M_s = M_s$ with $(I - A^+ A)$ idempotent—that permits this. Premultiplication of both sides of Eq. (46) shows this. At each instant of time, she thus determines the two orthogonal components $\ddot{q}_{\mathbb{N}(A)}$ and $\ddot{q}_{\mathbb{N}(A)^\perp}$ of the acceleration of the constrained system, and thus, its total acceleration $\ddot{q}$ (Eq. (11)).

In short, we see that at every instant of time Nature appears to behave like a calculating mathematician. She obtains the acceleration of every constrained mechanical system in two sequential steps. She first uses the minimum-norm least squares solution to find $\ddot{q}_{\mathbb{N}(A)^\perp} = A^+ b$ by solving the equation $A\ddot{q} = b$, which describes the constraints that the dynamical system is subjected to. She then uses an auxiliary unconstrained mechanical system described by Eq. (46) that has a special mass matrix M_s and again she uses the minimum-norm least squares solution to find $\ddot{q}_A$, which is the same as $\ddot{q}_{\mathbb{N}(A)}$.

2.3 Properties Associated With the Matrix $\tilde{M}$. Results 1 and 2 require that the matrix $\tilde{M} = [M_s | A^T]^T$ has full column rank. Recall that the matrix $M_s = (I - A^+ A) M (I - A^+ A)$. We now provide insights into this requirement and begin by showing that the rank of $\tilde{M}$ can usually be found more easily by considering the matrix $\hat{M} = [M | A^T]^T$ instead.

LEMMA 3. *The matrix $\tilde{M} = [M_s | A^T]^T$ has full column rank if and only if the matrix*

$$\hat{M} = [M | A^T]^T$$

has full column rank.

Proof. Part 1: We first show that if $\hat{M}$ does not have full column rank, then $\tilde{M}$ does not have full column rank. If $\hat{M}$ does not have full column rank, then there exists a nonzero n-vector w such that $\hat{M} w = [M | A^T]^T w = 0$, which implies that $Mw = 0$ and $Aw = 0$. Since the solution to the latter equation is $w = (I - A^+ A) u$, where u is an arbitrary n-vector, the former equation becomes $M(I - A^+ A) u = 0$. Premultiplying both sides of this equation by $(I - A^+ A)$, we find that there exists a vector u such that

$$0 = (I - A^+ A) M (I - A^+ A) u = (I - A^+ A) M (I - A^+ A)(I - A^+ A) u = M_s w \tag{48}$$

In the second equality, we have used the fact that $(I - A^+ A)$ is idempotent. Hence, there exists a non-zero n-vector w such that $M_s w = 0$ and $Aw = 0$, so that $[M_s | A^T]^T w = \tilde{M} w = 0$. Hence, $\tilde{M}$ does not have full column rank.

We have therefore shown that the proposition "$\tilde{M}$ has full column rank" which implies that $\hat{M}$ has full column rank.

Part 2: We next show that if $\tilde{M}$ does not have full column rank, then $\hat{M}$ does not have full column rank. If $\tilde{M}$ does not have full rank, then there exists a nonzero n-vector vector w such that $M_s w = 0$, and $Aw = 0$. Again, the latter equation has the solution $w = (I - A^+ A) u$, where u is an arbitrary n-vector, and the former equation now becomes

$$\begin{aligned} M_S w &= [(I - A^+ A) M (I - A^+ A)][(I - A^+ A) u] \\ &= (I - A^+ A) M (I - A^+ A) u = 0 \end{aligned} \tag{49}$$

Again, the idempotence of $(I - A^+ A)$ is used in the second equality. Premultiplying both sides of the last equality by u^T yields $[u^T (I - A^+ A) M^{1/2}][M^{1/2} (I - A^+ A) u] = \|M^{1/2} w\|^2 = 0$, which implies that $M^{1/2} w = 0$, so that $Mw = 0$. Hence, there exists an n-vector w such that $Mw = 0$ and $Aw = 0$.

We have shown that the proposition "$\hat{M}$ has full column rank" implies that $\tilde{M}$ has full column rank. Parts 1 and 2 imply the result. ■

We next present a result regarding the column rank of $\tilde{M}$ (Eq. (16)), which will be used in the following section where we obtain an algebraic expression for M_s^+ that appears in Eq. (24).

LEMMA 4. *The matrix $\tilde{M} = [M_s | A^T]^T$ has full column rank if and only if the symmetric matrix $\widehat{M} := V_2^T M V_2$ is non-singular (Eq. (43) for the definition of the matrix V_2).*

Proof. Part 1: We first show that if $\tilde{M}$ does not have full column rank, then $\widehat{M}$ is singular. If $\tilde{M}$ does not have full column rank, then there exists an n-vector vector $w \neq 0$ such that $\tilde{M} w = [M_s | A^T]^T w = 0$, which implies that

$$M_s w = 0, \text{ and } Aw = 0 \tag{50}$$

The solution to the second equation has the form $w = (I - A^+ A) z = V_2 (V_2^T z) := V_2 u \neq 0$, where u is some n-vector. Since $w \neq 0$, obviously $u \neq 0$, else w would be zero. Using this relation for w and Eq. (45) in the first equation in Eq. (50) gives

$$M_s w = V_2 \widehat{M} V_2^T w = V_2 \widehat{M} V_2^T V_2 u = V_2 \widehat{M} u = 0 \tag{51}$$

In the third equality, we have used the fact that $V_2^T V_2 = I$. Premultiplying both sides of the last equality by V_2^T gives $\widehat{M} u = 0$. Since u

$\neq 0$, $\widehat{M}$ is singular. We have therefore shown that the proposition "$\tilde{M}$ does not have full column rank" implies "$\widehat{M}$ is singular." In other words, we have shown that the proposition, "$\widehat{M}$ is non-singular," implies "$\tilde{M}$ has full column rank"

Part 2: We next prove the converse. We show that if $\widehat{M}$ is singular, then $\tilde{M}$ does not have full column rank. If $\widehat{M}$ is singular, then there exists some n-vector $u \neq 0$ such that $\widehat{M}u = 0$. Noting that $\widehat{M} = V_2^T M V_2$ and $V_2^T V_2 = I$, this equation can be rewritten as

$$V_2^T M V_2 u = V_2^T M V_2 \underbrace{V_2^T V_2}_{=I} u = 0 \tag{52}$$

Premultiplying both sides of the last equality in Eq. (52) by V_2, we get

$$V_2 V_2^T M V_2 V_2^T (V_2 u) = M_s(V_2 u) = 0 \tag{53}$$

which shows that there exists a nonzero vector $w = V_2 u \neq 0$ such that $M_s w = 0$. Since $u \neq 0$, the n-vector w cannot be zero. If it were zero, the columns of V_2 would have to be linearly dependent, which is not true because the columns of V_2 are orthonormal and therefore linearly independent.

We also know that $Aw = A(V_2 u) = 0$, since V_2 spans the null space of A. We have therefore shown that the proposition, "$\widehat{M}$ is singular," implies that "there exists an n-vector $w \neq 0$ such that $M_s w = 0$ and $Aw = 0$," i.e., "the matrix $\tilde{M}$ does not have full column rank." In other words, we have proved that the proposition, "$\tilde{M}$ has full column rank," implies, "$\widehat{M}$ is non-singular." ∎

Remark 6. Lemmas 3 and 4 show that when $M \geq 0$, the following three statements are equivalent: (i) $\tilde{M}$ has full column rank, (ii) $\tilde{M}$ has full column rank and (iii) $\widehat{M} = V_2^T M V_2$ is non-singular. ∎

2.4 Algebraic Properties of M_s^+

Lemma 5. *When the matrix $\tilde{M}$ has full column rank,*

(a) the Moore–Penrose(MP) inverse of $M_s = V_2 \widehat{M} V_2^T$ is given by $M_s^+ = V_2 \widehat{M}^{-1} V_2^T$ where $\widehat{M} := V_2^T M V_2$.

(b) $M_s M_s^+ = V_2 V_2^T = (I - A^+ A) = M_s^+ M_s$

Proof.

(a) By Lemma 4, the matrix $\widehat{M}$ is non-singular. Noting that $V_2^T V_2 = I$, we check that the four MP conditions are satisfied by M_s^+ as follows.

(1)

$$M_s M_s^+ M_s = V_2 \widehat{M} V_2^T V_2 \widehat{M}^{-1} V_2^T V_2 \widehat{M} V_2^T = V_2 \widehat{M} V_2^T = M_s$$

(2)

$$M_s^+ M_s M_s^+ = V_2 \widehat{M}^{-1} V_2^T V_2 \widehat{M} V_2^T V_2 \widehat{M}^{-1} V_2^T = V_2 \widehat{M}^{-1} V_2^T = M_s^+$$

(3) $M_s M_s^+ = V_2 \widehat{M} V_2^T V_2 \widehat{M}^{-1} V_2^T = V_2 V_2^T$, which is symmetric

(4) $M_s^+ M_s = V_2 \widehat{M}^{-1} V_2^T V_2 \widehat{M} V_2^T = V_2 V_2^T$, which is symmetric

(b) $M_s M_s^+ = V_2 \widehat{M} V_2^T V_2 \widehat{M}^{-1} V_2^T = V_2 V_2^T = (I - A^+ A)$.

The last equality in Part (b) of the Lemma is obvious, and $M_s M_s^+ = M_s^+ M_s$. ∎

We notice that the n-by-n matrix M_s^+ is obtained by inverting the non-singular, smaller $(n-k)$ by $(n-k)$ matrix $\widehat{M}$.

Results 1 and 2 along with Lemma 5 lead us to the following result.

Result 3. When $\tilde{M}$ has full column rank, the acceleration of the constrained mechanical system is explicitly given by

$$\ddot{q} = V_2 \widehat{M}^{-1} V_2^T (Q + C - MA^+ b) + A^+ b = V_2 \widehat{M}^{-1} V_2^T \Delta + A^+ b \tag{54}$$

and

$$\begin{aligned} Q^c &= M V_2 \widehat{M}^{-1} V_2^T (Q + C - MA^+ b) + MA^+ b - Q \\ &= M V_2 \widehat{M}^{-1} V_2^T \Delta + MA^+ b - Q \end{aligned} \tag{55}$$

where $\widehat{M} = V_2^T M V_2$ and $A = [U_1 | U_2] \Sigma [V_1 | V_2]^T$ using the notation in Eq. (43).

Proof. Using *Result 2* and Lemma 5 in Eq. (24) the result in Eq. (54) follows. Equation (55) follows from Eq. (31). ∎

3 Reduction of the New Equation of Motion to the Known Form of the FECM when M is Positive Definite

The equation of motion given in Result 1 allows the matrix M to be positive semidefinite. In this section, we show that the equation of motion obtained in Eq. (24) reduces to the Fundamental Equation of Constrained Motion that is known when $M > 0$ [11,12]. We first develop the following two Lemmas that will be required.

Lemma 6. *When $M > 0$, the $(m+n)$-by-n matrix*

$$\tilde{A} = \begin{bmatrix} (I - A^+ A) M^{1/2} \\ A M^{-1/2} \end{bmatrix} \tag{56}$$

has full column rank.

Proof. Noting that the positive definite matrices $M^{1/2}$ and $M^{-1/2}$ are non-singular, we have $rank[(I - A^+ A)M^{1/2}] = rank(I - A^+ A) = n - k$ and $rank(AM^{-1/2}) = rank(A) = k$. Also, since

$$\begin{aligned} [(I - A^+ A) M^{1/2}](AM^{-1/2})^T &= (I - A^+ A) M^{1/2} M^{-1/2} A^T \\ &= [A(I - A^+ A)]^T = 0 \end{aligned} \tag{57}$$

the rows of the matrix $(I - A^+ A)M^{1/2}$ are mutually orthogonal to those of $AM^{-1/2}$. Hence, the matrix $\tilde{A}$ has n independent rows, and its rank is n. ∎

Lemma 7. *When $M > 0$,*

$$I - M_s^+ M = M^{-1/2} B^+ A \tag{58}$$

where $B := AM^{-1/2}$.

Proof. We observe that the matrix $\tilde{A}$ defined in Eq. (56) satisfies both the requirements in Lemma 1 since, (a) by Eq. (57), $(I - A^+ A)M^{1/2}(AM^{-1/2})^T = 0$, and, (b) by Lemma 6, $\tilde{A}$ has full column rank.

Thus, according to Lemma 1

$$I = [(I - A^+ A)M^{1/2}]^+ [(I - A^+ A) M^{1/2}] + (AM^{-1/2})^+ (AM^{-1/2}) \tag{59}$$

which can be rewritten as

$$\begin{aligned} I &= M^{1/2}(I - A^+ A)[(I - A^+ A) M^{1/2} M^{1/2} (I - A^+ A)]^+ [(I - A^+ A) M^{1/2}] \\ &\quad + B^+ A M^{-1/2} \\ &= M^{1/2}(I - A^+ A)[M_s]^+ (I - A^+ A) M^{1/2} + B^+ A M^{-1/2} \\ &= M^{1/2} M_s^+ M^{1/2} + B^+ A M^{-1/2} \end{aligned} \tag{60}$$

In the first equality above, we have used the relation $X^+ = X^T (XX^T)^+$, in the second we have used Eq. (15), and in the third equality we have used Lemma 2, Part(b). Premultiplying both sides of Eq. (60) by $M^{-1/2}$ and postmultiplying both sides by $M^{1/2}$ give Eq. (58).

We note that postmultiplication of both sides of Eq. (58) by M^{-1} results in the relation

$$M_s^+ = M^{-1/2}(I - B^+ B) M^{-1/2} \tag{61}$$

which we will use shortly. ∎

We now consider the known Fundamental Equation of Constrained Motion when $M>0$ [11,12]. This equation says that an unconstrained system described by Eq. (1), which is subjected to a set of consistent constraints given by Eq. (5), along with a specification of the n-vector C given by Eq. (6), has its acceleration $\ddot{q}$ given explicitly by [Ref. 12, Eq. (39)]

$$\ddot{q} = a + M^{-1/2}B^{+}(b - Aa) + M^{-1/2}(I - B^{+}B)M^{-1/2}C \tag{62}$$

where $B := AM^{-1/2}$. We now show that when $M>0$, Eq. (24) reduces to Eq. (62).

Result 4

When $M>0$, Eq. (24) is the same as Eq. (62).

Proof. First, we observe that when $M>0$, then $\tilde{M}$ has full column rank. This follows from the fact that $M_s = [M^{1/2}(I - A^{+}A)]^{T} M^{1/2}(I - A^{+}A)$, so that

$$rank(M_s) = rank[M^{1/2}(I - A^{+}A)] = rank(I - A^{+}A) = n - k \tag{63}$$

But the rows of M_s are orthogonal to the rows of the matrix A (Eq. (25)), and A has rank k. This shows that $\tilde{M}$ has n independent rows. Since $rank(\tilde{M}) = n$, $\tilde{M}$ has full column rank.

Using *Result 1*, the acceleration of the constrained system is then provided by Eq. (24) (which we repeat here for convenience) as

$$\ddot{q} = M_s^{+}(Q + C - MA^{+}b) + A^{+}b \tag{64}$$

Equation (64) can be rearranged as

$$\begin{aligned}\ddot{q} &= M_s^{+}Q + (I - M_s^{+}M)A^{+}b + M_s^{+}C \\ &= M^{-1/2}(I - B^{+}B)M^{1/2}a + M^{-1/2}B^{+}AA^{+}b \\ &\quad + M^{-1/2}(I - B^{+}B)M^{-1/2}C \\ &= a - M^{-1/2}B^{+}Aa + M^{-1/2}B^{+}AA^{+}b + M^{-1/2}(I - B^{+}B)M^{-1/2}C \\ &= a + M^{-1/2}B^{+}(b - Aa) + M^{-1/2}(I - B^{+}B)M^{-1/2}C\end{aligned} \tag{65}$$

To get the second equality above, we have replaced M_s^{+} in the first and last member on the right-hand side (in the line above) using Eq. (61), Q by Ma, and $(I - M_s^{+}M)$ in the second member using Eq. (58). In the third equality, we have replaced B in the first member (in the line above) with $AM^{-1/2}$. In the last equality we have replaced $AA^{+}b$ by b, since a necessary and sufficient condition for the constraints to be consistent, i.e., the equation $A\ddot{x} = b$ to be consistent, is $AA^{+}b = b$. ■

4 Conclusions

This paper develops the equation of motion for general structural and mechanical systems with positive semidefinite mass matrices that are subjected to constraints that may be holonomic and/or nonholonomic. The constraints can be nonlinear in the generalized coordinates and velocities and need not be functionally independent. Systems that do not satisfy d'Alembert's assumption are included.

The equation of motion obtained shows the simple and elegant way in which Nature orchestrates the motion of constrained structural and mechanical systems. She appears to behave as a calculating mathematician would and obtains the orthogonal projections (components) of the acceleration of the constrained system in the null space of the constraint matrix A and in its orthogonal complement. The addition of these two components gives her the total acceleration of the constrained system. In his seminal paper on mechanics, which dealt with positive definite mass matrices, Gauss argues that Nature seems to behave like a mathematician [2] in dealing with the constrained motion of mechanical and structural systems. Our interpretation extends this argument to include a much broader—and, from a practical applications standpoint, an extremely useful—class of systems that have singular mass matrices. We show in detail the exact manner in which she accomplishes this.

Though the equations of motion for most structural and mechanical systems are nonlinear, we observe in this paper that Nature determines the acceleration of a constrained system at each instant of time by solving, somewhat surprisingly, *linear* equations and *always* using their *minimum-norm least squares solutions*. Since the minimum-norm least squares solution of any set of linear equations $Px = q$ is $x = P^{+}q$, where P^{+} is the (unique) Moore–Penrose inverse of the matrix P, this observation puts the notion of the "Moore–Penrose inverse" at the very center of analytical dynamics making it quintessential to our understanding of the motion of general constrained structural and mechanical systems.

Thinking along Gauss's lines [2], we conjecture that at each instant of time Nature appears to execute constrained motion in two sequential steps. First and foremost, she ensures that the acceleration of the constrained system satisfies the desired constraints. She accomplishes this by determining the minimum-norm least squares acceleration, $\ddot{q} = A^{+}b$, that satisfies the constraint equation $A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t)$. Geometrically, this solution is the orthogonal projection (component) of the acceleration $\ddot{q}$ on $\mathbb{R}(A^{T}) = \mathbb{N}(A)^{\perp}$. She next finds the component of $\ddot{q}$ that lies in the orthogonal complement of this space, namely, in $\mathbb{N}(A)$. To do this Nature, appears to use an auxiliary system with a special symmetric mass matrix M_s (Eq. (15)). She also uses a modified force that is obtained from the "given" force applied to the unconstrained system, by excluding from it the force generated by the acceleration $\ddot{q} = A^{+}b$, which she has already determined. This auxiliary equation she again solves using its minimum-norm least squares solution.

The paper also shows that the new equation developed here that gives the explicit acceleration of a structural or a mechanical system subjected to constraints when the mass matrix of the physical system is positive semidefinite reduces to the known fundamental equation of constrained motion for the case when the mass matrix is strictly positive definite.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

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